Announcements:

1) New webwork up, due Friday (supplement)

2) Exam 2 Monday

3) Practice problems for exam 2 on CTools under "Assignments"

Subspaces Associated to a Given Matrix

(Section 4.2)

Let A be an $m \times n$ matrix -or- think of A as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition: (kernel/nullspace)

The nullspace of A is the subspace of $\mathbb{R}^n$ consisting of those vectors v for which $\boxed{Av = 0}$.

Notation: $\mathrm{Nul}(A)$.

You might also see this called the kernel of A, denoted $\ker(A)$.

Given A, how do you find $\mathrm{Nul}(A)$?

Take the matrix

$$\begin{bmatrix} A & \vec{0} \end{bmatrix}$$

and row reduce it.

Example 1:

$$A = \begin{bmatrix} -1 & 3 & 5 & 6 \\ 2 & 0 & 4 & 18 \\ 10 & -8 & -2 & 0 \\ 0 & -8 & -22 & -90 \\ 0 & 6 & 14 & 30 \end{bmatrix}$$

Find Nul(A), then find a basis for it.

$$\text{rref}\begin{bmatrix} A & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix}$$

We get

$$\begin{bmatrix} 1 & 0 & 0 & -21 & 0 \\ 0 & 1 & 0 & -30 & 0 \\ 0 & 0 & 1 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This means

$$x_1 - 21x_4 = 0$$

$$x_2 - 30x_4 = 0$$

$$x_3 + 15x_4 = 0$$

So $x_1 = 21x_4$

$x_2 = 30x_4$

$x_3 = -15x_4$

We're solving

$$Ax = 0 \quad \text{where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x = \begin{bmatrix} 21x_4 \\ 30x_4 \\ -15x_4 \\ x_4 \end{bmatrix}$$

$$= x_4 \begin{bmatrix} 21 \\ 30 \\ -15 \\ 1 \end{bmatrix}$$

$$= \mathrm{Nul}(A)$$

This is a one-dimensional subspace with basis

$$\left\{ \begin{bmatrix} 21 \\ 30 \\ -15 \\ 1 \end{bmatrix} \right\}$$

Example 2:

Let $A = \begin{bmatrix} 2 & 1 & -5 \\ 4 & 2 & -10 \\ -14 & -7 & 35 \end{bmatrix}$

Find $\mathrm{Nul}(A)$ and a basis for it.

Solving $Ax = 0$

where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\text{rref}\begin{bmatrix} A & \begin{matrix}0\\0\\0\end{matrix} \end{bmatrix}$$

We get

$$\begin{bmatrix} 1 & 1/2 & -5/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This means

$$x_1 + \frac{x_2}{2} - \frac{5x_3}{2} = 0$$

Then

$$x_1 = -\frac{x_2}{2} + \frac{5x_3}{2}$$

$$\mathrm{Nul}(A) = \begin{bmatrix} -\frac{x_2}{2} + \frac{5x_3}{2} \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5/2 \\ 0 \\ 1 \end{bmatrix}$$

The dimension is 2, a basis is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Definition: (range)

The range of A is the subspace consisting of all vectors y such that there is a vector x with $\boxed{Ax = y}$

Denoted by $\mathrm{Ran}(A)$.

Note:

$\mathrm{Ran}(A) = \mathrm{Col}(A)$,

so we already

know how to

find it!

Definition: (row/column space)

The row space of a matrix A is the subspace formed by linear combinations of the rows of A. The column space of A is the subspace formed by linear combinations of the columns of A.

How do you find row/column spaces?

Take rref(A). The nonzero rows are basis for the row space.

For the column space, take rref(A^t); the nonzero rows are a basis for the column space.

Notation: Row(A) is the row space, Col(A) is the column space.

Example 3:

Let $A = \begin{bmatrix} 3 & 6 & 56 & 0 \\ -8 & 2 & 19 & 1 \\ -5 & 8 & 75 & 1 \end{bmatrix}$

Find Col(A), Row(A), and a basis for each.

Row(A): rref(A)

$$= \begin{bmatrix} 1 & 0 & -\frac{1}{27} & -\frac{1}{9} \\ 0 & 1 & \frac{505}{54} & \frac{1}{18} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Row(A) is then

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1/27 \\ 1/9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 505/54 \\ 1/18 \end{bmatrix} \right\}.$$

$$\text{Row}(A) = x_1 \begin{bmatrix} 1 \\ 0 \\ -1/27 \\ 1/9 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 505/54 \\ 1/18 \end{bmatrix}$$

for any real numbers x_1 and x_2.

For col(A),

$$\text{rref}(A^t)$$

$$= \text{rref}\left(\begin{bmatrix} 3 & -8 & -5 \\ 6 & 2 & 8 \\ 56 & 19 & 75 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis is then $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

The column space

is then

$$\mathrm{col}(A) = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

for any real numbers

x_1 and x_2.

Rank and Nullity

(Section 4.6)

Let A be an $m \times n$ matrix.

$$\text{Rank}(A) = \text{dimension of } \text{Ran}(A)$$
$$= \text{dimension of } \text{Col}(A)$$

$$\text{Nullity}(A) = \text{dimension of } \text{Nul}(A)$$

These are just numbers!

Rank Theorem

$$\boxed{\mathrm{Rank}(A) + \mathrm{Nullity}(A) = n.}$$

Application: (permutation matrices)